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The energetics of the interaction between short small-amplitude internal waves and inertial waves

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The interaction between a wave packet of small-amplitude short internal waves, and a finite-amplitude inertial wave field is described to second order in the short-wave amplitude. The discussion is based on the principle of wave action conservation and the equations for the wave-induced Lagrangian mean flow. It is demonstrated that as the short internal waves propagate through the inertial wave field they generate a wave-induced train of trailing inertial waves. The contribution of this wave-induced mean flow to the total energy balance is described. The results obtained here complement the finding of Broutman & Young (1986) that the short internal waves undergo a net change in energy after their encounter with the inertial wave field.

1. Introduction

The interaction between short and long waves is a topic of considerable interest and relevance to many branches of fluid mechanics. One technique for studying this interaction is the recently developed theory of wave–mean flow interaction, which in the fluid mechanics context, began with the pioneering work of Whitham (1965, 1970) and Bretherton & Garrett (1968), and culminated in the generalized Lagrangian-mean theory of Andrews & McIntyre (1978*a*, *b*); for a recent review see Grimshaw (1984). The key concept here is the notion of an averaging operator, which typically describes averages over either a short wave period or wavelength, and is used to distinguish between the short waves and the mean flow (i.e. the long waves). The effect of the mean flow on the waves is then described by the equation for conservation of wave action, while the effect of the waves on the mean flow is best described by the momentum equation for the Lagrangian mean flow in which the wave-induced forcing terms can be described either by the radiation stress tensor, or by the wave pseudomomentum. The explicit, detailed formulation can be found in Andrews & McIntyre (1978*a*, *b*), or Grimshaw (1984).

In this paper we propose to apply these ideas to the interaction between internal waves and inertial waves in the ocean. Thus the short-wave field is chosen to be a wave packet of small-amplitude internal waves, and the mean flow, or long-wave field, consists of finite-amplitude inertial waves. Our motivation for this study is the recent calculations by Broutman & Young (1986) of the trajectories of a wave packet of small-amplitude short internal waves which propagate vertically through the refracting current of a localized packet of large-scale inertial waves. Using ray theory for linearized waves they were able to calculate the net changes in frequency and vertical wavenumber of the short internal waves due to their encounter with the inertial wave field. In one of the more unexpected results, they found that there was a systematic tendency for the short internal waves to emerge from the inertial wave field with an increased frequency and decreased vertical wavenumber, contrary to the predictions of the induced diffusion approximation of weak interaction theory. Using the principle of wave action conservation they inferred that there was then a net increase in short-wave energy, which had been extracted from the inertial wave field. This process does not depend in any way on wave dissipation, and there would seem to be considerable practical as well as theoretical interest in studying it more closely, since it could operate effectively when observed mixing rates are low, as they typically are away from oceanic boundaries and the surface mixed layer.

Broutman & Young (1986) used the principle of wave action conservation to deduce changes in short-wave energy from their calculated changes in short-wave frequency. Although this is adequate as far as the short-wave field is concerned, it does not explain how the mean flow field supplies this energy change. This can only be elucidated by considering the wave-induced mean flow, and it is this aspect we address in this paper. If a is the amplitude of the short waves, we calculate the $O(|a|^2)$ wave-induced mean flow, and consider the $O(|a|^2)$ total energy balance. Using averages over the horizontal wavelength of the short waves, which is assumed to remain constant throughout the interaction, we shall show that as the short internal waves pass through the inertial wave field, they excite an $O(|a|^2)$ wave-induced train of inertial waves which has a phase velocity different from the original, pre-existing inertial waves. After the short internal waves have passed through the inertial wave field, this trailing wave-induced inertial wavetrain remains behind, and, together with the original inertial wave field, contributes to the $O(|a|^2)$ total energy balance. This $O(|a|^2)$ change in the mean flow field exactly balances the $O(|a|^2)$ change in the short-wave energy. In §2 we describe the formulation of the wave action equation and the wave-induced mean flow equation in the present context, and in §3 we present our results.

Before proceeding to our discussion it is pertinent to mention two related studies. Andrews (1980) describes the mean motion induced by vertically propagating short internal-inertial waves, which are generated by a horizontal corrugated boundary which propagates horizontally. His study differs from ours in two important respects. First, the short waves are propagating into a medium at rest (i.e. there is no basic mean flow), and secondly, the wave-induced mean flow field is supported by a horizontal pressure gradient (in the direction transverse to the moving boundary). In the case we shall discuss there are no horizontal mean pressure gradients as we assume that the short-wave field is homogeneous and periodic in both horizontal directions. Hence the original three-dimensional problem is effectively reduced to one in which the significant wave and mean field variables vary only with respect to the vertical coordinate and the time. Hasselmann (1970) calculated the wave-induced mean flow due to standing internal-inertial waves in the ocean (i.e. the short-wave field has a modal representation in the vertical, and propagates only in the horizontal direction). Like the case considered here, he found that for horizontally homogeneous conditions, the wave-induced mean flow is an inertial wave, which is, however, a standing wave in his formulation. There are, of course, a number of calculations of wave-induced mean flows excited by internal waves, in a context where the influence of the earth's rotation is ignored (see, for instance, Grimshaw 1984).

2. Formulation

The wave packet is defined by an amplitude $a(\boldsymbol{x},t)$ and a phase $\theta(\boldsymbol{x},t)$ such that a typical field variable such as the vertical particle displacement is described by

$$\mathrm{Re}\,[a(\boldsymbol{x},t)\exp\{\mathrm{i}\theta(\boldsymbol{x},t)\}]. \tag{2.1}$$

The local frequency $\omega(\boldsymbol{x},t)$ and wavenumber vector $\boldsymbol{\kappa}(\boldsymbol{x},t)$ are then defined by

$$\omega = -\theta_t,\ \boldsymbol{\kappa} = \nabla\theta, \tag{2.2}$$

and satisfy the local dispersion relation for internal gravity waves. For an inviscid, incompressible fluid this is

$$\hat{\omega}^2 = \frac{N^2\kappa_{\mathrm{H}}^2 + f^2m^2}{\kappa_{\mathrm{H}}^2+m^2}, \tag{2.3a}$$

where

$$\hat{\omega} = \omega - \boldsymbol{\kappa}_{\mathrm{H}}\cdot\boldsymbol{u}_0. \tag{2.3b}$$

Here $\kappa_{\mathrm{H}} = |\boldsymbol{\kappa}_{\mathrm{H}}|$ where $\boldsymbol{\kappa}_{\mathrm{H}}$ is the horizontal component of $\boldsymbol{\kappa}$, m is the vertical component of $\boldsymbol{\kappa}, N(z)$ is the buoyancy frequency, f is the constant inertial frequency and $\boldsymbol{u}_0$ is the basic horizontal velocity field. The amplitude a, frequency ω and wavenumber vector $\boldsymbol{\kappa}$ are slowly varying functions of $\boldsymbol{x}, t$, and in contrast the phase θ is a rapidly varying function. The dispersion relation (2.3*a*) can be regarded as a partial differential equation for the phase θ. Alternatively we can eliminate θ from (2.2) to obtain the equations

$$\boldsymbol{\kappa}_t + \nabla\omega = 0. \tag{2.4}$$

The amplitude is determined from the equation for the conservation of wave action (Grimshaw 1975)

$$A_t + \nabla\cdot\{(\boldsymbol{u}_0 + \boldsymbol{V})A\} = 0, \tag{2.5a}$$

where

$$\boldsymbol{V} = \nabla_{\kappa}\hat{\omega}, \tag{2.5b}$$

$$A = E\hat{\omega}^{-1}, \tag{2.5c}$$

and

$$E = \tfrac{1}{2}\rho_0|a|^2\left(N^2 + \frac{f^2m^2}{\kappa_{\mathrm{H}}^2}\right). \tag{2.5d}$$

Here $\boldsymbol{V}$ is the intrinsic group velocity, A is the wave action density, E is the wave energy density, and $\rho_0(z)$ is the basic density field.

Mean flow quantities are denoted by an overbar, which here denotes a Lagrangian average and is obtained by averaging over the phase θ. In the application to follow, the horizontal wavenumber vector $\boldsymbol{\kappa}_{\mathrm{H}}$ will be constant, and the averaging operation is then equivalent to averaging over a horizontal wavelength. The wave-induced Lagrangian mean flow $\bar{\boldsymbol{u}}$ is given by (Grimshaw 1975, 1984; Andrews & McIntyre 1978*a*)

$$\tilde{\rho}\left\{\frac{\mathrm{d}\bar{\boldsymbol{u}}}{\mathrm{d}t} + f\boldsymbol{k}\times\bar{\boldsymbol{u}} + g\boldsymbol{k}\right\} + \nabla\bar{p} = -\bar{\nabla}\cdot\boldsymbol{R}, \tag{2.6}$$

where $\boldsymbol{R}$ is the radiation stress tensor, $\bar{p}$ is the Lagrangian mean pressure and $\tilde{\rho}$ is a mean density which satisfies the kinematic equation

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} + \tilde{\rho}\nabla\cdot\bar{\boldsymbol{u}} = 0. \tag{2.7}$$

The mean-flow equations are closed by the incompressibility condition which here takes the form,

$$\frac{\mathrm{d}\bar{J}}{\mathrm{d}t}+\bar{J}\nabla\cdot\bar{\boldsymbol{u}}=0, \tag{2.8}$$

where J is the Jacobian of the mapping from a Lagrangian field point to an Eulerian field point, and is a mean quantity (i.e. $J=\bar{J}$). Here the convective derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\bar{\boldsymbol{u}}\cdot\nabla, \tag{2.9}$$

so that the wave-induced forcing terms are accounted for by the radiation stress tensor $\boldsymbol{R}$ and $\bar{J}$. $\boldsymbol{R}$ is $O(|a|^2)$, and its dominant term is $\boldsymbol{V\kappa}A$; the remaining terms are either diagonal terms which contribute an $O(|a|^2)$ correction to the mean pressure, or terms which contribute an $O(|a|^2)$ correction to the mean density $\tilde{\rho}$. Also $\bar{J}$ equals 1 plus an $O(|a|^2)$ slowly varying correction term which can be ignored in the sequel.

An alternative and equivalent formulation of the mean-flow equations is one in which the wave-induced forcing terms involve the pseudomomentum (or quasimomentum) (Grimshaw 1975; Andrews & McIntyre 1978*a*). For an inviscid, incompressible fluid, this is given by

$$\bar{\rho}\left\{\frac{\mathrm{d}\bar{\boldsymbol{u}}}{\mathrm{d}t}+f\boldsymbol{k}\times\bar{\boldsymbol{u}}+g\boldsymbol{k}\right\}+\nabla\bar{p}=\frac{\partial\boldsymbol{P}}{\partial t}+(\bar{\boldsymbol{u}}\cdot\nabla)\,\boldsymbol{P}+\nabla\bar{\boldsymbol{u}}\cdot\boldsymbol{P}+\bar{\rho}\nabla Q, \tag{2.10}$$

where $\bar{\rho}$ is the Lagrangian mean density, and satisfies the equation

$$\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}t}=0. \tag{2.11}$$

The equations are again closed by (2.8), and we note that $\tilde{\rho}=\bar{\rho}\bar{J}$. The wave-induced forcing terms are now the pseudomomentum

$$\boldsymbol{P}=\boldsymbol{\kappa}A, \tag{2.12}$$

and Q which is an $O(|a|^2)$ quantity and provides $O(|a|^2)$ corrections to the mean pressure $\bar{p}$ and the mean density $\bar{\rho}$. It is given by

$$Q=\tfrac{1}{4}N^2|a|^2. \tag{2.13}$$

The Eulerian mean velocity is $\bar{\boldsymbol{u}}_{\mathrm{E}}=\bar{\boldsymbol{u}}-\bar{\boldsymbol{u}}_{\mathrm{S}}$ where $\bar{\boldsymbol{u}}_{\mathrm{S}}$ is the Stokes velocity, given by (Grimshaw 1975)

$$\bar{\boldsymbol{u}}_{\mathrm{S}}=-\boldsymbol{\kappa}\times\nabla\left\{\frac{Afm}{\rho_0\,\hat{\omega}(\kappa_{\mathrm{H}}^2+m^2)}\right\}. \tag{2.14}$$

$\bar{\boldsymbol{u}}_{\mathrm{S}}$ is $O(|a|^2)$, but since the wave packet is slowly varying is generally smaller than $\bar{\boldsymbol{u}}$. To the order considered it is also non-divergent, and since $\bar{\boldsymbol{u}}_{\mathrm{E}}$ is non-divergent, this confirms that we can effectively put $\bar{J}=1$ in (2.8). Note that $\bar{\boldsymbol{u}}_{\mathrm{S}}$ bears no simple relationship to the pseudomomentum, such as the simple equality which sometimes holds for irrotational flows (see Andrews & McIntyre 1978*a* for further discussion on this point).

We now restrict attention to the case when all wave variables (i.e. ω, $\boldsymbol{\kappa}$ and a) and all mean-flow variables are functions of z, t alone. First we observe that the basic horizontal velocity field $\boldsymbol{u}_0(z,t)$ is then given by

$$\boldsymbol{u}_{0t} + f\boldsymbol{k} \times \boldsymbol{u}_0 = 0. \tag{2.15}$$

The solution of this equation is an inertial wave packet, given by

$$u_0 + \mathrm{i}v_0 = F(z) \exp\left\{-\mathrm{i}f\left(t - \frac{z}{c}\right)\right\}, \tag{2.16}$$

where (u_0, v_0) are the (x,y)-components respectively of $\boldsymbol{u}_0$. The constant c is the vertical phase speed of the inertial wave, and $F(z)$ is an amplitude envelope which may be specified arbitrarily. Equation (2.4) now implies that $\boldsymbol{\kappa}_\mathrm{H}$ is a constant; note that with no loss of generality we may put $\boldsymbol{\kappa}_\mathrm{H} = (k, 0)$ so that the x-axis is aligned with the horizontal component of the wavenumber vector of the wave field. Equation (2.4) reduces to

$$m_t + \omega_z = 0, \tag{2.17}$$

where $\omega = \omega(m; z, t)$ is defined by (2.3a, b). This equation can be solved by ray methods, and some typical solutions are described by Broutman (1984) and Broutman & Young (1986). The wave action equation reduces to

$$A_t + (WA)_z = 0, \tag{2.18a}$$

where

$$W = \frac{\partial \hat{\omega}}{\partial m} = -\frac{m\kappa_\mathrm{H}^2(N^2 - f^2)}{\hat{\omega}(\kappa_\mathrm{H}^2 + m^2)^2}. \tag{2.18b}$$

Here W is the vertical group velocity. Once m has been found from (2.17), this equation is readily solved by ray methods.

The forcing terms in the mean flow equations, or (2.10), are now known. We simplify these equations by putting

$$\bar{\boldsymbol{u}} = \boldsymbol{u}_0 + \bar{\boldsymbol{u}}_2, \tag{2.19}$$

assuming that $\bar{\boldsymbol{u}}_2$ is $O(|a|^2)$, and retaining only $O(|a|^2)$ terms. We find that

$$\rho_0\{\bar{\boldsymbol{u}}_{2t} + f\boldsymbol{k} \times \bar{\boldsymbol{u}}_2\} = \boldsymbol{\kappa}_\mathrm{H} A_t. \tag{2.20}$$

If we let $(\bar{u}_2, \bar{v}_2)$ be the (x,y)-components of $\bar{\boldsymbol{u}}_2$, and (k, l) are likewise the components of $\boldsymbol{\kappa}_\mathrm{H}$, then the solution of (2.20) is

$$\rho_0(\bar{u}_2 + \mathrm{i}\bar{v}_2) = (k + \mathrm{i}l)\, U, \tag{2.21a}$$

where

$$U = \int_{-\infty}^{t} A_s(s,z) \exp\{-\mathrm{i}f(t-s)\}\, \mathrm{d}s. \tag{2.21b}$$

Here we are assuming that at each level z, as $t \to -\infty$, no waves are present, and hence there is no wave-induced mean flow as $t \to -\infty$. Further (2.21b) can be expressed in the form

$$U = A(t,z) + \hat{U}, \tag{2.22a}$$

where

$$\hat{U} = -\mathrm{i}f \int_{-\infty}^{t} A(s,z) \exp\{-\mathrm{i}f(t-s)\}\, \mathrm{d}s. \tag{2.22b}$$

The first term in (2.22a) represents an instantaneous wave-induced mean flow $\boldsymbol{\kappa}_\mathrm{H} A$, equal to the wave pseudomomentum, which is associated locally with the wave packet. The second term $\hat{U}$ describes a train of forced inertial waves. Explicit illustrations and more detailed analysis will be taken up in §3.

Next we derive the total energy equation for the wave-induced components of the flow. First, using (2.5*a*, *c*) it may be shown that

$$E_t + (WE)_z + \frac{EW}{\hat{\omega}}\boldsymbol{\kappa}_{\mathrm{H}} \cdot \boldsymbol{u}_{0z} = 0, \tag{2.23}$$

where we recall that E (equation (2.5*d*)) is the wave energy density. Note that the last term in (2.23) is the product of the radiation stress tensor with the basic flow velocity gradient, and describes the transfer of energy between the waves and the mean flow. The $O(|a|^2)$ term in the total energy is

$$\mathscr{E} = E + \rho_0 \boldsymbol{u}_0 \cdot \bar{\boldsymbol{u}}_2. \tag{2.24}$$

Using this definition, (2.15), (2.20) and (2.23), it may be shown that

$$\mathscr{E}_t + (\omega W A)_z = 0. \tag{2.25}$$

Hence the total energy is conserved, and the vertical flux of the total energy is $\omega W A$. Next we note that

$$\rho_0 \boldsymbol{u}_0 \cdot \bar{\boldsymbol{u}}_2 = \boldsymbol{\kappa}_{\mathrm{H}} \cdot \boldsymbol{u}_0 A(t, z) + \int_{-\infty}^{t} f \boldsymbol{\kappa} \times \boldsymbol{u}_0 \cdot \boldsymbol{\kappa}_{\mathrm{H}} A(s, z)\, \mathrm{d}s. \tag{2.26}$$

From (2.24) we see that $\rho_0 \boldsymbol{u}_0 \cdot \bar{\boldsymbol{u}}_2$ is the $O(|a|^2)$ contribution of the wave-induced mean flow to the total energy. Interestingly the integrand of the second term is the product of the Coriolis term in the basic flow with the wave pseudomomentum $\boldsymbol{\kappa}_{\mathrm{H}} A$. Further, using (2.5*c*) and (2.26), we see that (2.24) becomes

$$\mathscr{E} = \hat{\mathscr{E}} + \int_{-\infty}^{t} f \boldsymbol{k} \times \boldsymbol{u}_0 \cdot \boldsymbol{\kappa}_{\mathrm{H}} A(s, z)\, \mathrm{d}s, \tag{2.27a}$$

where

$$\hat{\mathscr{E}} = \omega A. \tag{2.27b}$$

The first term in (2.27*a*) is $\hat{\mathscr{E}}$, here equal to the pseudoenergy (Andrews & McIntyre 1978*b*, or Grimshaw 1984), and represents a local term associated with the wave packet. The second term is the contribution of the train of forced inertial waves, and is non-local. Finally we present the equation for the pseudoenergy itself. From (2.17) and (2.18*a*) this is

$$\hat{\mathscr{E}}_t + (W\hat{\mathscr{E}})_z + \hat{\mathscr{E}}_\omega \boldsymbol{\kappa}_{\mathrm{H}} \cdot \boldsymbol{u}_{0t} = 0. \tag{2.28}$$

Note that $\hat{\mathscr{E}}$ is conserved only when the basic flow is time-independent, which for the present case requires that $f = 0$ (see (2.16)). Also note that the vertical flux of total energy is here equal to $W\hat{\mathscr{E}}$, which is just the vertical flux of pseudoenergy.

3. Results

The process of determining the wave-induced mean flow has been reduced to solving the following sequence of equations. First (2.17) is combined with the dispersion relation (2.3*a*, *b*) to determine ω, m and hence the vertical group velocity W. Then (2.18*a*) is solved to determine the wave action density A. Finally the wave-induced mean flow $\bar{\boldsymbol{u}}_2$ is evaluated from (2.21*a*), by evaluating the expression (2.22*a*) for U. For simplicity, we shall suppose that the buoyancy frequency N is a constant.

We proceed to illustrate the general theory by considering four cases of increasing complexity.

3.1. *Non-rotating* ($f = 0$)

Although this case is well known (see, for instance, the review by Grimshaw 1984), we present it here for completeness. With $f = 0$ it follows from (2.15) that $\boldsymbol{u}_0 = \boldsymbol{u}_0(z)$ is independent of t. Further $\hat{U}$ (equation (2.22b)) is zero, and the wave-induced mean flow $\rho_0 \bar{\boldsymbol{u}}_2$ is here just equal to $\boldsymbol{\kappa}_{\mathrm{H}} A$, the wave pseudomomentum, which is always locally associated with the wave packet. Further, from (2.27a) we see that the total energy is just the pseudoenergy ωA. The essential features of this case are captured by considering the solution of (2.17) for which $\omega = \omega_0$, a constant. Then the dispersion relation (2.3a, b) determines $m = m(z)$ from the equation

$$\hat{\omega}(m) + \boldsymbol{\kappa}_{\mathrm{H}} \cdot \boldsymbol{u}_0(z) = \omega_0. \tag{3.1}$$

The vertical group velocity $W = W(z)$ can then also be found, and the solution of the wave action equation (2.18a) is given by

$$WA = \alpha(t - \phi(z)), \tag{3.2a}$$

where

$$\phi(z) = \int^z W^{-1}\,\mathrm{d}z. \tag{3.2b}$$

The equation $t = \phi(z)$ describes the ray trajectories. Note that

$$\oint A\,\mathrm{d}z = \text{constant}, \tag{3.3}$$

where the integral is taken over the wave packet. This result is readily obtained directly from (2.18a), and in this instance can also be attributed to total energy conservation (see (2.25)). It shows that the wave packet length varies inversely as the wave action density. However, the corresponding result for the wave energy density is

$$\frac{\partial}{\partial t} \oint E\,\mathrm{d}z = -\oint WE \frac{\boldsymbol{\kappa}_{\mathrm{H}} \cdot \boldsymbol{u}_{0z}}{\hat{\omega}}\,\mathrm{d}z, \tag{3.4}$$

and the right-hand side describes the interaction of the wave packet with the mean flow. Indeed the change in kinetic energy of the mean flow due to the waves is

$$\rho_0 \boldsymbol{u}_0 \cdot \bar{\boldsymbol{u}}_2 = \boldsymbol{\kappa}_{\mathrm{H}} \cdot \boldsymbol{u}_0 A,$$

and

$$\frac{\partial}{\partial t} \oint \rho_0 \boldsymbol{u}_0 \cdot \bar{\boldsymbol{u}}_2 = \oint WE \frac{\boldsymbol{\kappa}_{\mathrm{H}} \cdot \boldsymbol{u}_{0z}}{\hat{\omega}}\,\mathrm{d}z. \tag{3.5}$$

Of course, the sum of (3.4) and (3.5) is just an expression of the conservation of total energy (see (2.25)).

Two scenarios can now be identified. Suppose, without loss of generality that $W > 0$ and $\boldsymbol{\kappa}_{\mathrm{H}} \cdot \boldsymbol{u}_{0z} > 0$. If $\hat{\omega} > 0, m < 0$ and the interaction is of the critical level type, since as the wave propagates upwards, $\hat{\omega}$ decreases towards zero, $|m|$ increases towards infinity and W decreases. Eventually the wave packet is absorbed at the critical level (defined by $\hat{\omega} = 0$). As the wave packet propagates upwards it loses energy to the mean flow (see (3.4) and (3.5)). Alternatively $\hat{\omega} < 0, m > 0$ and the interaction is of the turning-point type, since as the wave propagates upwards, $\hat{\omega}$ increases towards N and $|m|$ decreases towards zero. Eventually the wave packet is reflected at the turning point, $|\hat{\omega}| = N$. As the wave propagates upwards it gains energy from the mean flow (see (3.4) and (3.5)).

3.2. *Zero basic flow* ($\boldsymbol{u}_0 = \mathbf{0}$)

In this case $F = 0$ in (2.16), but we shall assume that $f \neq 0$. The essential features of this case can be exhibited by considering the solution of (2.15) for which m (and hence $\boldsymbol{\kappa}$) and ω are constants, satisfying the dispersion relation (2.3*a*). The vertical group velocity W is then also a constant, and the solution of the wave action equation is

$$A = \beta(t - W^{-1}z), \tag{3.6}$$

which describes a wave packet. The wave-induced mean flow is given by (2.21*a, b*) and (2.22*a*) where the non-local term $\hat{U}$ is given by

$$\hat{U} = -\mathrm{i}f \exp(-\mathrm{i}f\tau) \int_{-\infty}^{\tau} \beta(u) \exp(\mathrm{i}fu)\, \mathrm{d}u, \tag{3.7a}$$

where
$$\tau = t - W^{-1}z. \tag{3.7b}$$

Clearly (3.7*a*) describes a train of forced inertial waves, whose vertical phase speed is W. These forced inertial waves are generated upon arrival of the wave packet (i.e. $\hat{U} \to 0$ as $t \to -\infty$), but remain after the wave packet has passed, since, although now free inertial waves, they have zero vertical group velocity. Indeed

$$\hat{U} \to -\mathrm{i}f \exp(-\mathrm{i}f\tau) \int_{-\infty}^{\infty} \beta(u) \exp(\mathrm{i}fu)\, \mathrm{d}u, \quad \text{as } t \to \infty, \tag{3.8}$$

where we are implicitly assuming that the wave packet has finite extent. Note that, at the point of initiation, the forced inertial waves are $\frac{1}{2}\pi$ out of phase with the wave packet.

3.3. *Periodic inertial wave*

In this case we shall suppose that F in (2.16) is a constant, $F = F_0$ say, so that the basic inertial wave is a periodic wave, with phase speed c. It is then appropriate to seek a solution of (2.4) for which ω and m are functions only of

$$z' = z - ct, \tag{3.9}$$

and hence are steady in a frame of reference moving with the inertial wave. This case has been discussed in detail by Broutman & Young (1986), although we shall repeat the essential results here for completeness. The solution of (2.4) is then

$$\omega' = \omega - mc = \hat{\omega} + \boldsymbol{k}_{\mathrm{H}} \cdot \boldsymbol{u}_0 - mc = \omega_0', \tag{3.10}$$

where ω_0' is a constant. This relation, together with the local dispersion relation (2.3*a*), then defines $m = m(z')$, and hence the vertical group velocity $W = W(z')$. Let us consider the case of greatest oceanographic interest when $c > 0$ and $\omega > 0$. The inertial wave packet has upward phase propagation, but downward group velocity, while the internal waves have downward phase propagation but upward group velocity. For the internal waves $W > 0$ requires that $\hat{\omega}m < 0$, where we are assuming that $N^2 > f^2$. The graph of $\hat{\omega} - mc$ as a function of m is shown in figure 1. There are two cases to consider depending on whether $c \gtrless W_{\mathrm{M}}$, where W_{M} is the maximum value of the group velocity, and is defined by

$$W_{\mathrm{M}} = |W(m = m_{\mathrm{M}})|, \tag{3.11a}$$

where
$$\frac{3m_{\mathrm{M}}^2 f^2}{\kappa_{\mathrm{H}}^2} = -(N^2 + f^2) + \{(N^2+f^2)^2 + 3N^2f^2\}^{\frac{1}{2}}. \tag{3.11b}$$

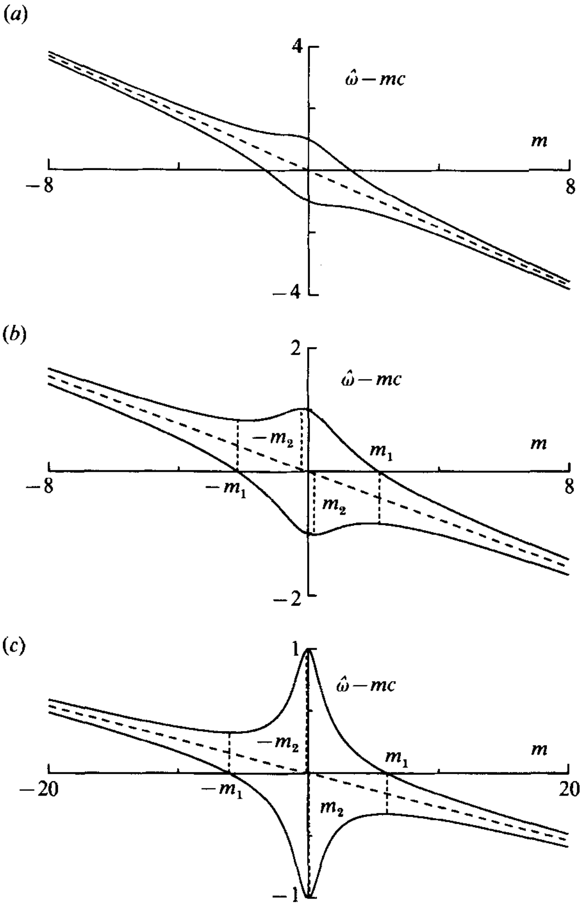

FIGURE 1. The graph of $\hat{\omega}-mc$ as a function of m. Note that each curve has slope $W' = W-c$ and each asymptotes to the (dashed) line $-mc$. In (*a*) $c = 1.2W_M$, so there are no turning points. In (*b*) $c = 0.5W_M$, and the turning points (where $W' = 0$) are marked by $\pm m_{1,2}$. As $c \to 0$, the high-frequency low-wavenumber turning point m_2 approaches zero. For example, (*c*) shows this case for the smaller value of $c = 0.07W_M$, closer to the parameter range considered by Broutman & Young (1986). In all graphs the vertical axis is non-dimensionalized by N, and the horizontal axis by κ_H. The computations are made with $N/f = 75$.

Note that for $N^2 \gg f^2$, $m_M^2 \approx \frac{1}{2}\kappa_H^2$ and $W_M \approx 2N/3\sqrt{3}\kappa_H$. For $c > W_M$, the graph of $\hat{\omega}-mc$ has the shape shown in figure 1(*a*) and has no turning points, since the relative group velocity $W' = W-c < 0$ for all m. It follows that m is a single-valued function of $\boldsymbol{\kappa}_H \cdot \boldsymbol{u}_0(z')$ and the ray paths contain no caustics. In this quasi-steady case the ray path is periodic, due to the periodic nature of $\boldsymbol{u}_0$ (see (2.16)). For $c < W_M$ the graph of $\hat{\omega}-mc$ has the shape shown in figures 1(*b*) and 1(*c*) and has two turning points $(\pm m_{1,2})$ on each branch, where $W' = 0$. Depending on the value of $\boldsymbol{\kappa}_H \cdot \boldsymbol{u}_0(z')$ it is now possible for m to be multi-valued, and the ray paths may contain caustics whenever $W' = 0$. The ray paths are again periodic. Note that the lower (upper) branch is used when $m > 0 (< 0)$. We also note that in the mid-frequency approximation discussed by Broutman & Young (1986) $m^2 \gg m_M^2$ and $c \ll W_M$ so that only the case of figure 1(*c*) applies, and caustics are generated only by the larger turning point $|m| = m_1$.

The solution of the wave action equation (2.18*a*) is now given by

$$W'A = \gamma(t - \psi(z')), \tag{3.12a}$$

where
$$\psi(z') = \int^{z'} (W')^{-1}\,\mathrm{d}z' \tag{3.12b}$$

The equation $t = \psi(z')$ describes the ray paths. Note that, as in §3.1, the result (3.3) again holds. The solution (3.12*a*) fails at caustics where W' tends to zero as the square root of the distance from the caustic. This singularity can be removed using Airy function approximations (Broutman 1986). The wave-induced flow is now given by (2.21*a, b*) and (2.22*a, b*) where here the non-local term $\hat{U}$ is given by

$$\hat{U} = -\frac{\mathrm{i}f}{c}\exp\left(\frac{\mathrm{i}fz'}{c}\right)\int_{z'}^{\infty}\gamma\left(\frac{(z-u)}{c} - \psi(u)\right)\exp\left(-\frac{\mathrm{i}fu}{c}\right)\frac{\mathrm{d}u}{W'(u)}. \tag{3.13}$$

As in §3.2, (3.13) describes a train of forced inertial waves which are generated upon arrival of the wave packet (i.e. $\hat{U} \to 0$ as $z' \to \infty$), but remain after the wave packet has passed, since

$$\hat{U} \to -\frac{\mathrm{i}f}{c}\exp\left(\frac{\mathrm{i}fz'}{c}\right)G(z), \quad \text{as } t \to \infty, \tag{3.14a}$$

where
$$G(z) = \int_{-\infty}^{\infty}\gamma\left(\frac{(z-u)}{c} - \psi(u)\right)\exp\left(-\frac{\mathrm{i}fu}{c}\right)\frac{\mathrm{d}u}{W'(u)}. \tag{3.14b}$$

Comparing (3.14*a*) with (2.16) we see that (3.14*a*) describes an inertial wave, although it cannot necessarily be deduced that the vertical phase speed is c, since, as the case of §3.2 shows, the phase of $G(z)$ in general contributes to the vertical phase speed. Indeed, if v is the vertical phase speed, then

$$v^{-1} = c^{-1} + f^{-1}\,\mathrm{Im}\left(\frac{1}{G}\frac{\mathrm{d}G}{\mathrm{d}z}\right). \tag{3.15}$$

Note that although the integrand of $G(z)$ (equation (3.14*b*)) contains a singularity where W' vanishes, the singularity is of the square root kind, and integrable, so that $G(z)$ is finite.

The energetics of the interaction of the wave packet with the mean flow can either be succinctly described by (3.3), or by (2.25) for total energy conservation. However, the non-local nature of $\bar{\boldsymbol{u}}_2$ together with the periodic and non-local nature of $\boldsymbol{u}_0$, prevent us from obtaining a local result for the total energy analogous to (3.3) for the wave action density, since global integrals of $\rho_0 \boldsymbol{u}_0 \cdot \bar{\boldsymbol{u}}_2$ (see (2.24)) are divergent. The energetics of the interaction between an internal wave packet and an inertial wave is best discussed within the context of §3.4.

3.4. *Inertial wave packet*

In this case we shall suppose that $F(z)$ in (2.16) has finite extent (i.e. F is zero for $|z|$ sufficiently large), and hence (2.16) describes an inertial wave packet. The internal wave packet, also of finite extent, propagates through the inertial wave packet, passing from one uniform region where $\boldsymbol{u}_0 \equiv \boldsymbol{0}$ to another. The situation is described schematically in figure 2. In the uniform region the internal wave packet is described in §3.2, and has constant values of m and ω, satisfying the dispersion relation (2.3*a*); note that when $\boldsymbol{u}_0 \equiv \boldsymbol{0}$, $\hat{\omega} = \omega$. We let m_b and ω_b denote the values of m and ω before the internal wave ray encounters the inertial wave packet, and m_a and ω_a be the

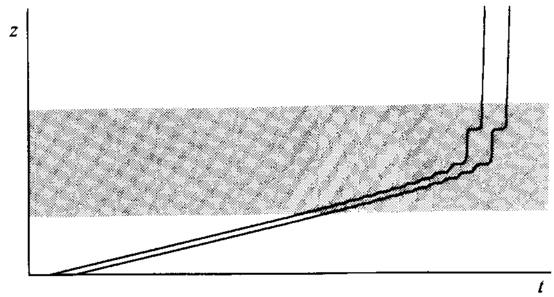

FIGURE 2. A schematic plot of the trajectory of the internal wave packet as it propagates through the inertial wave packet. The shaded region denotes the vertical extent of the inertial wave packet, and the two rays shown mark the outermost edges of the internal wave packet.

corresponding values after the encounter. The values m_b and ω_b can be regarded as the same constants for all the internal wave rays which make up the internal wave packet, but m_a and ω_a in general depend on which ray is being considered. Broutman & Young (1986) have reported typical ray calculations, with parameter values in the mid-frequency range for which $m^2 \gg m_M^2 \approx \frac{1}{2}\kappa_H^2$ and $c \ll W_M$. In this case the ray trajectories pass through a number of caustics located within the inertial wave packet where $W = c$, $W' = 0$ and $|m| = m_1$ (see figure 1 (*b*, *c*)). Of course, here m varies with both z and t as the internal wave ray passes through the inertial wave, although some progress can be made in understanding the situation by supposing that $F(z)$ in (2.16) is a slowly varying function relative to the phase (i.e. $|F'(z)/F(z)| \ll f/c$), and hence ω' (3.10) is also a slowly varying function. Within this approximation m_a and ω_a are approximately independent of the particular ray being considered. An alternative approximation which leads to the same result is to assume that the internal wave packet is narrow-banded. Broutman & Young (1986) have exploited the slowly varying approximation in analysing their numerical results for ray trajectories. They have shown that in general m_a can be markedly different from m_b, particularly when m_b is substantially different from m_1. They have also demonstrated that there is a tendency for $|m_a| < |m_b|$ and hence $|\omega_a| > \omega_b|$, essentially because the internal wave spends more time within the inertial wave packet with $|m| > m_1$ and low group velocity ($W < c$ or $W' < 0$), than with $|m| < m_1$ and large group velocity ($W > c$, or $W' < 0$). The change-over occurs at caustics ($|m| = m_1$ or $W' = 0$), and the internal wave is more likely to escape from the inertial wave packet when it has a large group velocity. We shall not give further details as our main purpose here is to explore the energetics associated with the change in wavenumber and frequency.

The energetics of the interaction is described either by the wave action equation (2.18*a*), or equation (2.25) for total energy conservation. We shall examine both as they give complementary views of the energy exchange. First, from the wave action equation (2.18*a*) it again follows that (3.3) holds. We shall use the slowly varying approximation so that we may regard m_a and ω_a as constant across the wave packet after it has emerged from the inertial wave. Then, recalling that the wave energy density $E = \hat{\omega}A$ and $\hat{\omega} = \omega$ when $\boldsymbol{u}_0 = \mathbf{0}$, it follows that

$$\frac{\left(\oint E\,\mathrm{d}z\right)_b}{\omega_b} \approx \frac{\left(\oint E\,\mathrm{d}z\right)_a}{\omega_a} \tag{3.16}$$

where the subscripts indicate the values before and after the internal wave encounters the inertial wave, and the integrals are taken over the internal wave packet. Thus the ratio ω_a/ω_b determines whether the internal wave packet gains or loses energy during its traverse of the inertial wave. Since ω_a/ω_b is usually greater than one, as explained above (see Broutman & Young 1986), it follows that there is a tendency for the internal wave packet to gain energy.

Next we observe from (2.25) for total energy conservation that

$$\int_{-\infty}^{\infty} \mathscr{E}\,\mathrm{d}z = \text{constant}. \tag{3.17}$$

Using (2.24) it follows that

$$\left(\oint E\,\mathrm{d}z\right)_{\mathrm{a}} - \left(\oint E\,\mathrm{d}z\right)_{\mathrm{b}} = -\int_{-\infty}^{\infty} (\rho_0\,\boldsymbol{u}_0\cdot\bar{\boldsymbol{u}}_2)_{\mathrm{a}}\,\mathrm{d}z, \tag{3.18}$$

where we note that $\rho_0\,\boldsymbol{u}_0\cdot\bar{\boldsymbol{u}}_2$, the $O(|a|^2)$ contribution of the wave-induced mean flow to the total energy, is given by (2.26), and is zero before the interaction but is non-zero afterwards due to the non-local nature of the wave-induced mean flow. This, of course, is given by (2.21*a, b*) and (2.22*a, b*), and the non-local term $\hat{U}$ describes a train of forced inertial waves. Equation (3.18) shows that the change in energy of the internal wave packet is directly due to the wave-induced forced inertial waves, and the consequent change in the kinetic energy of the mean flow $\bar{\boldsymbol{u}}$, which to $O(|a|^2)$ is given by the basic flow $\boldsymbol{u}_0$ plus the wave-induced component $\bar{\boldsymbol{u}}_2$, and thus consists of the superposition of two systems of inertial waves. Next, using (2.22*a, b*) and (2.26) it may be shown that

$$\begin{aligned} -\int_{-\infty}^{\infty} (\rho_0\,\boldsymbol{u}_0\cdot\bar{\boldsymbol{u}}_2)_{\mathrm{a}}\,\mathrm{d}z &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\boldsymbol{k}\times\boldsymbol{u}_0\cdot\boldsymbol{\kappa}_{\mathrm{H}}\,A\,\mathrm{d}t\,\mathrm{d}z, \\ &= -\mathrm{Re}\left\{\mathrm{i}f(k-\mathrm{i}l)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(t,z)\,F(z)\exp\left\{\mathrm{i}f\left(\frac{z}{c}-t\right)\right\}\mathrm{d}t\,\mathrm{d}z\right\}. \end{aligned} \tag{3.19}$$

The integration is over the intersection of the internal wave packet trajectory and the inertial wave packet (see figure 2). Note that although the ray paths may contain caustics where A is singular, the singularities are integrable. The expression (3.19) is not as useful as (3.16), as it is not immediately obvious how to determine the sign of (3.19). However, we can demonstrate that (3.18) is equivalent to (3.16) as follows. If a typical ray is given by $z = z(t, z_0)$ where z_0 is a label for each ray (for instance, z_0 is the initial value of z on each ray so that $z(0, z_0) = z_0$), then the solution of the wave action equation (2.18*a*) is

$$A(z,t)\frac{\partial z}{\partial z_0} = A_0(z_0), \tag{3.20}$$

where $A_0(z_0)$ is a constant on each ray, and hence can be evaluated as $t \to -\infty$. Note that the result (3.3) can now be expressed in the form

$$\oint A\,\mathrm{d}z = \oint A_0(z_0)\,\mathrm{d}z_0, \tag{3.21}$$

from which (3.16) follows immediately. Next we change variables in (3.19) to z_0 and t, so that the right-hand side of (3.19) becomes

$$-\int_{-\infty}^{\infty} A_0(z_0)\left(\int_{-\infty}^{\infty} \{f\boldsymbol{k}\times\boldsymbol{u}_0\cdot\boldsymbol{\kappa}_{\mathrm{H}}\}_{z=z(t,z_0)}\,\mathrm{d}t\right)\mathrm{d}z_0. \tag{3.22}$$

But, (2.15) implies that

$$\frac{\partial\omega}{\partial t}+W\frac{\partial\omega}{\partial z}=\boldsymbol{\kappa}_{\mathrm{H}}\cdot\boldsymbol{u}_{0t}=-f\boldsymbol{k}\times\boldsymbol{u}_0\cdot\boldsymbol{\kappa}_{\mathrm{H}}, \tag{3.23}$$

where the left-hand side is the derivative of ω along a ray. Thus (3.22) becomes

$$\int_{-\infty}^{\infty} A_0(z_0)\,(\omega_{\mathrm{a}}-\omega_{\mathrm{b}})\,\mathrm{d}z_0. \tag{3.24}$$

Then using the slowly varying approximation, in which ω_{a}, as well as ω_{b}, is constant across the wave packet, it follows that (3.24) becomes

$$\frac{\left(\oint E\,\mathrm{d}z\right)_{\mathrm{b}}}{\omega_{\mathrm{b}}}(\omega_{\mathrm{a}}-\omega_{\mathrm{b}}), \tag{3.25}$$

where we have used the limit $t\rightarrow-\infty$ to evaluate the right-hand side of (3.21). With this result we see that (3.18) is equivalent to (3.16).

To summarize, we have shown that as a small-amplitude internal wave packet propagates through a large-amplitude inertial wave field it excites a wave-induced mean flow which consists of two parts. One part is the wave pseudomomentum which is localized to the internal wave packet. The other part is not localized and is a wave-induced train of trailing inertial waves, which is distinct from the pre-existing inertial waves. Broutman & Young (1986) have demonstrated that after the internal wave packet has passed through the inertial wave field, there is a net change in internal wave energy. Here we have shown that this change in energy is accounted for by the wave-induced generation of trailing inertial waves.

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